

NON-EXISTENCE OF ETERNAL SOLUTIONS TO LAGRANGIAN MEAN CURVATURE FLOW

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ABSTRACT. In this paper, we derive a mean curvature estimate for eternal solutions (including translating solutions) of almost-calibrated Lagrangian mean curvature flow in complex Euclidean space. As a consequence, we show a non-existence result for eternal solutions of almost-calibrated Lagrangian mean curvature flow.

1. INTRODUCTION

Eternal solutions are solutions of mean curvature flow defined on a whole time interval $(-\infty, \infty)$. Since mean curvature flow is quasi-linear parabolic equation, we cannot solve the flow backward in time. Hence eternal solutions are very special and have interesting properties if they exist. Translating solutions are solutions of the mean curvature flow which have translation invariance under mean curvature flow, and they are typical examples of eternal solutions.

Eternal solutions are type II singularity models of mean curvature flow, that is, they naturally arise after the parabolic rescaling at type II singularities. In codimension one, R. Hamilton [4] showed that any complete convex eternal solution whose curvature attains its maximum at a point in space-time must be a translating solution. However this fact is still not clear in higher codimension. Therefore, it is natural to consider not only translating solutions, but also eternal solutions to study type II singularities.

Besides above facts, Chen-Li [2] and M.-T. Wang [22] independently showed that type I singularities do not appear under almost-calibrated (see section 2) Lagrangian mean curvature flow. Later Neves [13] generalize this result to the zero-Maslov class case. In this case, therefore, we essentially need to investigate type II singularities and eternal solutions (including translating solutions) as the singularity models.

Our aim in this paper is to prove a non-existence theorem for eternal solutions of almost-calibrated Lagrangian mean curvature flow. In this direction, we know some results by Han-Sun [6], Neves-Tian [14] and Sun [20]. However, all the known results mention only translating solutions. In [6], Han and Sun showed a non-existence result of almost-calibrated Lagrangian translating solitons with nonnegative sectional curvature. We generalize their theorem [6] into the class of Lagrangian eternal solutions under a similar situation.

Date: October 24, 2016, preprint.

2010 Mathematics Subject Classification. Primary 53C44; Secondary 35C06.

Key words and phrases. Mean curvature flow, eternal solutions, Bernstein type theorem.

The author is supported by the Grant-in-Aid for JSPS Fellows.

Theorem 1.1. *There is no non-flat complete Lagrangian eternal solution with nonnegative Ricci curvature to the almost-calibrated Lagrangian mean curvature flow in \mathbb{C}^n with $\cos \theta \geq \delta > 0$ for each time $t \in (-\infty, \infty)$.*

In general, the blow-up limit of type II singularity must be non-flat. Therefore, as a consequence of our result, we obtain the following.

Corollary 1.2. *Any eternal solution with nonnegative Ricci curvature cannot arise as blow-up limit of almost-calibrated Lagrangian mean curvature flow in \mathbb{C}^n .*

Our result is a parabolic version of Theorem by Han and Sun [6], and the proof is somewhat similar to [6]. However, to show the parabolic version of the curvature estimate, we adopt another technique by Souplet-Zhang [15] and M. Wang [21] to show a parabolic version of curvature estimate. Thanks to the parabolic curvature estimate, we do not need to assume the translation symmetry which is needed in [6].

Acknowledgement. The author is supported by the Grant-in-Aid for JSPS Fellows. During the preparation of this paper the author has stayed in Max Planck Institute for Mathematics in the Sciences, Leipzig. The author is grateful to Jürgen Jost for his hospitality and his interest. The author also would like to thank Reiko Miyaoka for her helpful comments.

2. PRELIMINARIES

2.1. Lagrangian submanifold in \mathbb{C}^n . In the following, our setting follows [13] and [14]. Let J and $\omega = \langle J(\cdot), \cdot \rangle$, respectively, be the standard complex structure on \mathbb{C}^n , $n \geq 1$ and the standard symplectic form on \mathbb{C}^n . We consider the closed complex valued n -form given by

$$\Omega := dz^1 \wedge \cdots \wedge dz^n,$$

where $z^j = x^j + \sqrt{-1}y^j$, $1 \leq j \leq n$, are complex coordinates of $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

A smooth n -dimensional submanifold L^n in \mathbb{C}^n is said to be Lagrangian if $\omega|_L \equiv 0$. This implies that

$$\Omega|_L = e^{\sqrt{-1}\theta} \text{vol}_L,$$

where vol_L denotes the volume form of L and θ is some multivalued function called *Lagrangian angle* or *phase function*. When the Lagrangian angle is single valued, the Lagrangian is called *zero-Maslov* class, and if

$$\cos \theta > 0,$$

then the Lagrangian is said to be *almost-calibrated*.

We denote by $A(\cdot, \cdot)$ the second fundamental form, and $\vec{H} := \text{trace} A$ the mean curvature vector of the submanifold L in \mathbb{C}^n . Now, by the Gauss equation, we can write the scalar curvature Scal_L of L as follows:

$$\text{Scal}_L = |\vec{H}|^2 - |A|^2.$$

Therefore, nonnegative Ricci curvature of L implies

$$(2.1) \quad |A|^2 \leq |\vec{H}|^2.$$

Note also that the important relation between the Lagrangian angle θ and the mean curvature vector \vec{H} is given by

$$(2.2) \quad \vec{H} = J\nabla\theta.$$

As an immediate consequence of this, we easily obtain the following inequality:

$$(2.3) \quad |\nabla \cos \theta|^2 \leq |\vec{H}|^2.$$

2.2. Lagrangian mean curvature flow. Next, we consider a deformation of a Lagrangian submanifold L^n in \mathbb{C}^n . Let $F_0 : L^n \rightarrow \mathbb{C}^n$ be an immersion with the second fundamental form A . The mean curvature flow is a one parameter family of smooth immersions $F : L^n \times [0, t_{\max}) \rightarrow \mathbb{C}^n$ which satisfies the following:

$$\begin{cases} \frac{\partial F}{\partial t}(p, t) = \vec{H}(p, t), & p \in L^n, t \geq 0, \\ F(\cdot, 0) = F_0. \end{cases}$$

We denote by $L_t := F_t(L) := F(L, t)$ a time slice of the flow $(L_t)_{t \geq 0}$. Note that mean curvature flow can be considered in more general situation. The submanifold is not necessarily Lagrangian and the ambient space is not necessarily Euclidean. However, it is shown by Smoczyk in [16] that the Lagrangian condition is preserved under mean curvature flow in Kähler-Einstein manifolds. In this case we call the flow *Lagrangian mean curvature flow*. It is also known that the zero-Maslov class condition is preserved under the Lagrangian mean curvature flow in Calabi-Yau manifolds. Moreover, according to [17], the Lagrangian angle θ can be chosen so that

$$\partial_t \theta = \Delta \theta,$$

and this implies

$$(2.4) \quad (\partial_t - \Delta) \cos \theta = |\vec{H}|^2 \cos \theta,$$

where ∂_t denotes the partial derivative with respect to t . Therefore, almost-calibrated condition is also preserved under the Lagrangian mean curvature flow in Calabi-Yau manifolds by the parabolic maximum principle. It is worth nothing that there exist no closed almost-calibrated Lagrangian submanifolds in \mathbb{C}^n . Hence, our attention is in only the case of non-compact complete almost-calibrated Lagrangian submanifolds in \mathbb{C}^n .

We note here another important evolution equation for later use (see for example [2], [19]):

$$(2.5) \quad (\partial_t - \Delta) |\vec{H}|^2 = -2|\nabla^\perp \vec{H}|^2 + 2|P|^2,$$

where ∇^\perp denotes the connection on the normal bundle of L and $P(\cdot, \cdot) := \langle A(\cdot, \cdot), \vec{H} \rangle$, therefore

$$|P|^2 \leq |A|^2 |\vec{H}|^2.$$

2.3. Eternal solutions. A solution of the mean curvature flow is called *eternal* if it is defined for $t \in (-\infty, \infty)$. This kind of solution is arise as a blow-up limit at a type II singularity of mean curvature flow if the initial submanifold is closed. Note that the blow-up limit is not necessarily compact. Indeed, any eternal solution in \mathbb{C}^n must be non-compact. We call a solution *Lagrangian eternal solution* if it is eternal and each time-slice is Lagrangian. All minimal submanifolds are stationary solution of the mean curvature flow, thus they are examples of eternal solutions.

A submanifold L^n in \mathbb{C}^n is called *translating soliton* or *translator* if there exist a nonzero constant vector $V \in \mathbb{C}^n$ so that

$$(2.6) \quad \vec{H} = V^N,$$

where V^N is the normal part of V to the submanifold L^n . Let $L_0 \subset \mathbb{C}^n$ be a translating soliton and consider the following deformation of L_0 :

$$(2.7) \quad L_t := L_0 + tV,$$

It is clear that this deformation (L_t) is merely translation of the initial submanifold L_0 in the constant direction of V , so that it is defined for $t \in (-\infty, \infty)$. Furthermore we can show that (2.7) satisfies mean curvature flow. Conversely, it is known that if the solution of mean curvature flow can be written as the form (2.7) for some submanifold L_0 and a constant vector $V \in \mathbb{C}^n$, then $L_0 \subset \mathbb{C}^n$ must satisfy (2.6), that is, a translating soliton. A solution of mean curvature flow which has the form (2.7) is called *translating solution*. Of course, translating solutions are eternal solutions. By the translation symmetry, the study of a translating solution (a parabolic equation) is reduced to the study of a translating soliton (an elliptic equation) which is each time slice of the translating solution.

Remark 2.1. Some people call (2.7) “translating soliton”.

Example 2.2 (grim reaper in \mathbb{C}). The curve defined by

$$\gamma_t := \{(-\log \cos y + t, y) \mid -\pi/2 < y < \pi/2, t \in (-\infty, \infty)\}$$

is called the *grim reaper* and it is a translating solution with the direction $V = (1, 0)$. It is known that the only translating solution in $\mathbb{C} \cong \mathbb{R}^2$ is the grim reaper.

Example 2.3 (hairclip in \mathbb{C} , see [7]). There is an eternal solution in \mathbb{C} given by

$$\xi_t := \{(x(t), y(t)) \mid \sinh x(t) = e^{-t} \cos y(t), t \in (-\infty, \infty)\},$$

which is not a translating solution. It is called *hairclip* and it looks like infinitely many grim reapers, alternating between translating up and translating down for $t \rightarrow -\infty$, and converges to a vertical line as $t \rightarrow \infty$.

Example 2.4 (eternal solution in \mathbb{C}^2). Let $c_t \subset \mathbb{C}$ be an eternal solution. Then a Lagrangian surface given by

$$L_t := c_t \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C}$$

is also Lagrangian eternal solution.

Note that any curve in \mathbb{C} is trivially Lagrangian and Ricci-flat. In this case the Lagrangian angle θ is the angle between the tangent vector of the curve and x -axis. Hence, above three eternal solutions give examples of Ricci-flat almost-calibrated Lagrangian eternal solutions. However, they all satisfy

$$\inf_{L_t} \cos \theta = 0, \quad \forall t \in (-\infty, \infty).$$

Therefore, the assumption $\cos \theta \geq \delta > 0$ is needed in our main theorem.

3. MEAN CURVATURE ESTIMATE

In this section we show a mean curvature estimate. As a consequence, we prove the main theorem 1.1. Our calculation is really similar to gradient estimate in [15] and [21] for harmonic map heat flow on complete manifolds. Using their technique, the author showed a parabolic Bernstein type theorem for graphic eternal solutions of mean curvature flow (in codimension one) with bounded slope in [10]. For graphic eternal solutions, the key point is to use the quantities $|A|^2$ and the slope of the graph to show the parabolic curvature estimate. In our current case, that is, for almost-calibrated Lagrangian eternal solution, it turns out that $|\vec{H}|^2$ and Lagrangian angle $\cos \theta$ match for the parabolic curvature estimate.

3.1. Mean curvature estimate on a cylindrical domain. Let $(L_t)_{t \in [-T, T]}$ be a complete solution to almost-calibrated Lagrangian mean curvature flow with $L_0 = L$, a complete Lagrangian submanifold in \mathbb{C}^n . We denote by $r(p) := d(o, p)$ the intrinsic distance on L from a fixed point $o \in L$ to a point $p \in L$. Of course if $p \in L$ is not in cut locus of $o \in L$, the distance function r is differentiable and $|\nabla r| = 1$. Let $D_R = D_R(o)$ be a closed intrinsic distance ball with center o and radius $R > 0$. Moreover, take a space-time cylindrical domain $D_{R,T} = D_{R,T}(o) := D_R(o) \times [-T, T]$. Define the function $\varphi(p, t) := 1 - \cos \theta(p, t)$ on $(L_t)_t$. Now, our mean curvature estimate on the cylindrical domain is given as follows.

Theorem 3.1. *Let $F : L^n \times [-T, T] \rightarrow \mathbb{C}^n$ be a complete solution to almost-calibrated Lagrangian mean curvature flow which has nonnegative Ricci curvature. Assume that there exists a positive constant $\delta > 0$ such that $\cos \theta(p, t) \geq \delta > 0$ for any point in $L^n \times [-T, T]$. Then there exists a constant C which is independent of R and T such that*

$$\sup_{D_{R/2, T/2}} \frac{|\vec{H}|}{b - \varphi} \leq C \left(\frac{1}{R} + \frac{1}{\sqrt{R}} + \frac{1}{\sqrt{T}} \right),$$

where b is a constant such that $\sup_{L_t} \varphi \leq 1 - \delta < b < 1, \forall t \in [-T, T]$.

For a complete eternal solution (L_t) , there exists a global constant b such that $\sup_{L_t} \varphi \leq 1 - \delta < b < 1, \forall t \in (-\infty, \infty)$. Take $R \rightarrow \infty$ and $T \rightarrow \infty$ in Theorem 3.1, then it follows $|\vec{H}| \rightarrow 0$. Finally, we obtain $|A| \equiv 0$ under the assumption $|A|^2 \leq |\vec{H}|^2$.

Corollary 3.2 (Theorem 1.1). *There is no non-flat complete Lagrangian eternal solution with nonnegative Ricci curvature to the almost-calibrated Lagrangian mean curvature flow in \mathbb{C}^n with $\cos \theta \geq \delta > 0$ for each time $t \in (-\infty, \infty)$.*

Remark 3.3. We adopt the intrinsic distance ball in this paper. On the other hand, we used extrinsic ball for the curvature estimate of codimension one entire graphic eternal solutions in [10]. Therefore, there are slight differences in the proof between this paper and [10].

3.2. Proof of the theorem. We prove Theorem 3.1 here. We assume that there exists a positive constant $\delta > 0$ so that

$$\cos \theta \geq \delta > 0.$$

Set $\varphi := 1 - \cos \theta$. Then since $\sup \varphi \leq 1 - \delta$, there exists a constant b such that

$$\sup_{L_t} \varphi \leq 1 - \delta < b < 1, \quad \forall t \in [-T, T].$$

We also assume that (L_t) has nonnegative Ricci curvature for $t \in [-T, T]$. Therefore the inequality (2.1): $|A|^2 \leq |\vec{H}|^2$ always holds. Combining (2.1) and (2.5) we obtain

$$(3.1) \quad (\partial_t - \Delta)|\vec{H}|^2 \leq -2|\nabla^\perp \vec{H}|^2 + 2|\vec{H}|^4.$$

Define the function on the eternal solution $(L_t)_{t \in [-T, T]}$ by

$$\phi = \frac{|\vec{H}|^2}{(b - \varphi)^2}.$$

A direct calculation shows that

$$(3.2) \quad \nabla \phi = \frac{\nabla |\vec{H}|^2}{(b - \varphi)^2} + \frac{2|\vec{H}|^2 \nabla \varphi}{(b - \varphi)^3}.$$

Similarly we can compute

$$(3.3) \quad \Delta \phi = \frac{\Delta |\vec{H}|^2}{(b - \varphi)^2} + \frac{4\langle \nabla \varphi, \nabla |\vec{H}|^2 \rangle}{(b - \varphi)^3} + \frac{2|\vec{H}|^2 \Delta \varphi}{(b - \varphi)^3} + \frac{6|\nabla \varphi|^2 |\vec{H}|^2}{(b - \varphi)^4}.$$

On the other hand, the time derivative of ϕ is given by

$$(3.4) \quad \partial_t \phi = \frac{\partial_t |\vec{H}|^2}{(b - \varphi)^2} - \frac{2|\vec{H}|^2 \partial_t \cos \theta}{(b - \varphi)^3}.$$

Subtracting (3.4) from (3.3), we obtain

$$\begin{aligned} (\Delta - \partial_t)\phi &= \frac{\Delta |\vec{H}|^2 - \partial_t |\vec{H}|^2}{(b - \varphi)^2} + \frac{4\langle \nabla \varphi, \nabla |\vec{H}|^2 \rangle}{(b - \varphi)^3} \\ &\quad + \frac{2|\vec{H}|^2 \partial_t \cos \theta - 2|\vec{H}|^2 \Delta \cos \theta}{(b - \varphi)^3} + \frac{6|\nabla \varphi|^2 |\vec{H}|^2}{(b - \varphi)^4}. \end{aligned}$$

By using (2.4) and (3.1), we compute

$$\begin{aligned} (\Delta - \partial_t)\phi &\geq \frac{2|\nabla^\perp \vec{H}|^2 - 2|\vec{H}|^4}{(b - \varphi)^2} + \frac{4\langle \nabla \varphi, \nabla |\vec{H}|^2 \rangle}{(b - \varphi)^3} \\ &\quad + \frac{4|\vec{H}|^4 \cos \theta}{(b - \varphi)^3} + \frac{6|\nabla \varphi|^2 |\vec{H}|^2}{(b - \varphi)^4}. \end{aligned}$$

Note that the following relations hold,

$$(3.5) \quad \frac{2|\nabla^\perp \vec{H}|^2}{(b - \varphi)^2} + \frac{2|\nabla \varphi|^2 |\vec{H}|^2}{(b - \varphi)^4} \geq \frac{4|\nabla^\perp \vec{H}| |\nabla \varphi| |\vec{H}|}{(b - \varphi)^3},$$

$$(3.6) \quad \frac{2\langle \nabla |\vec{H}|^2, \nabla \varphi \rangle}{(b - \varphi)^3} + \frac{4|\vec{H}|^2 |\nabla \varphi|^2}{(b - \varphi)^4} = \frac{2\langle \nabla \varphi, \nabla \phi \rangle}{(b - \varphi)}.$$

By using (3.5) and (3.6) with the Cauchy-Schwarz inequality and Kato's inequality ($|\nabla |\vec{H}|^2| \leq |\nabla^\perp \vec{H}|^2$), we finally obtain

$$(3.7) \quad (\Delta - \partial_t)\phi \geq \frac{2(1 - b)|\vec{H}|^4}{(b - \varphi)^3} + \frac{2\langle \nabla \varphi, \nabla \phi \rangle}{b - \varphi}.$$

Remark 3.4. The parabolic inequality (3.7) has actually the same form as the corresponding one in [10].

Now we take a smooth function $\eta(r, t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ supported on $[-R, R] \times [-T, T]$ which has the following properties:

1. $\eta(r, t) \equiv 1$ on $[-R/2, R/2] \times [-T/2, T/2]$ and $0 \leq \eta \leq 1$,
2. $\eta(r, t)$ is decreasing if $r \geq 0$, i.e., $\partial_r \eta \leq 0$,
3. $|\partial_r \eta|/\eta^a \leq C_a/R$, $|\partial_r^2 \eta|/\eta^a \leq C_a/R^2$ when $0 < a < 1$,
4. $|\partial_t \eta|/\eta^a \leq C_a/T$ when $0 < a < 1$.

Such a function can be given by a canonical way (for example, see [10], [12], [15] or [21]). We use a cut-off function supported on $D_{R,T}$ given by $\psi(p, t) := \eta(r(p), t)$.

Let $\mathcal{L} := -2\nabla\varphi/(b - \varphi)$. By using (3.7) we can calculate

$$\begin{aligned}
 (3.8) \quad & \Delta(\psi\phi) + \langle \mathcal{L}, \nabla(\psi\phi) \rangle - 2 \left\langle \frac{\nabla\psi}{\psi}, \nabla(\psi\phi) \right\rangle - \partial_t(\psi\phi) \\
 &= \psi(\Delta\phi - \partial_t\phi) + \phi(\Delta\psi - \partial_t\psi) + \langle \psi\mathcal{L}, \nabla\phi \rangle + \langle \phi\mathcal{L}, \nabla\psi \rangle - 2 \frac{|\nabla\psi|^2}{\psi} \phi \\
 &\geq 2\psi \frac{(1-b)|\vec{H}|^4}{(b-\varphi)^3} + \phi(\Delta\psi - \partial_t\psi) - 2 \frac{\langle \nabla\varphi, \nabla\psi \rangle}{b-\varphi} \phi - 2 \frac{|\nabla\psi|^2}{\psi} \phi.
 \end{aligned}$$

Note that $D_{R,T}$ is compact. Hence $\psi\phi$ attains its maximum at some point (p_0, t_0) in $D_{R,T}$. We may assume that such a point $p_0 \in L$ is not in the cut locus of $o \in L$ (see [3], [15] or [21]). At the point $(p_0, t_0) \in D_{R,T}$, we have

$$\nabla(\psi\phi) = 0, \quad \Delta(\psi\phi) \leq 0, \quad \partial_t(\psi\phi) \geq 0.$$

Hence by (3.8), we obtain

$$\begin{aligned}
 2\psi(1-b) \frac{|\vec{H}|^4}{(b-\varphi)^3} &\leq 2\phi \frac{\langle \nabla\varphi, \nabla\psi \rangle}{b-\varphi} + 2\phi \frac{|\nabla\psi|^2}{\psi} + \phi(\partial_t\psi - \Delta\psi) \\
 &= I + II + III.
 \end{aligned}$$

Note that the following holds:

$$|\nabla\psi|^2 = |\partial_r \eta|^2 |\nabla r|^2 \leq |\partial_r \eta|^2.$$

Recall that the following Young's inequality:

Lemma 3.5 (Young's inequality). *For any $a, b > 0$, and any $\varepsilon > 0$, we have*

$$ab \leq \varepsilon a^p + \varepsilon^{-\frac{p}{p-1}} b^q,$$

where $1 < p, q < \infty, 1/p + 1/q = 1$.

By using (2.3), Young's inequality and the property of η , we can estimate I as follows:

$$\begin{aligned}
 (3.9) \quad & I \leq 2\phi \frac{|\nabla\varphi|}{b-\varphi} |\nabla\psi| \leq 2\phi \frac{|\vec{H}|}{b-\varphi} |\nabla\psi| = 2\phi^{\frac{3}{2}} |\nabla\psi| \\
 & \leq \frac{\varepsilon}{4} \psi \phi^2 + \frac{C(\varepsilon) |\nabla\psi|^4}{\psi^3} \leq \frac{\varepsilon}{4} \psi \phi^2 + \frac{C(\varepsilon) |\partial_r \eta|^4}{\psi^3} \\
 & \leq \frac{\varepsilon}{4} \psi \phi^2 + \frac{C(\varepsilon)}{R^4},
 \end{aligned}$$

where $\varepsilon > 0$ is an arbitrary constant which is determined later, $C(\varepsilon)$ are constants depending only on ε . Similarly, we can calculate by using Young's inequality and the property of η ,

$$(3.10) \quad II = 2\phi \frac{|\nabla \psi|^2}{\psi} \leq \frac{\varepsilon}{4} \psi \phi^2 + C(\varepsilon) \frac{|\nabla \psi|^4}{\psi^3} \leq \frac{\varepsilon}{4} \psi \phi^2 + \frac{C(\varepsilon)}{R^4}.$$

Since the Ricci curvature of L is nonnegative, by the Laplacian comparison theorem we have

$$(3.11) \quad \Delta r \leq \frac{n-1}{r}.$$

Using (3.11) and $\partial_r \eta \leq 0$, we have

$$\Delta \psi = (\Delta r)(\partial_r \eta) + |\nabla r|^2(\partial_r^2 \eta) \geq \frac{n-1}{r}(\partial_r \eta) - |\partial_r^2 \eta|.$$

Hence we obtain for the second term of III in the same way as above,

$$(3.12) \quad \begin{aligned} -\phi \Delta \psi &\leq \phi |\partial_r^2 \eta| + \frac{(n-1)|\partial_r \eta|}{r} \phi \\ &\leq \phi |\partial_r^2 \eta| + \frac{2(n-1)|\partial_r \eta|}{R} \phi \\ &\leq \frac{\varepsilon}{4} \psi \phi^2 + C(\varepsilon, n) \left(\frac{1}{R^4} + \frac{1}{R^2} \right), \end{aligned}$$

where $C(\varepsilon, n)$ is a constant depending only on ε and n . (Note that we may assume $R/2 \leq r$ for the second inequality since $\partial_r \eta \equiv 0$ for $r \leq R/2$.)

As for the first term of III , we have

$$(3.13) \quad \begin{aligned} \phi(\partial_t \psi) &\leq \phi |\partial_t \eta| \leq \frac{\varepsilon}{4} \psi \phi^2 + C(\varepsilon) \frac{|\partial_t \eta|^2}{\psi} \\ &\leq \frac{\varepsilon}{4} \psi \phi^2 + \frac{C(\varepsilon)}{T^2}. \end{aligned}$$

From (3.9), (3.10), (3.12) and (3.13), we finally obtain

$$2(1-b)(b-\varphi)\psi\phi^2 \leq \varepsilon\psi\phi^2 + C(\varepsilon, n) \left(\frac{1}{R^4} + \frac{1}{R^2} + \frac{1}{T^2} \right).$$

Since $\varepsilon > 0$ is arbitrary we can take a sufficiently small ε such that

$$2(1-b)(b-\varphi) - \varepsilon > 0.$$

Then we have

$$(\psi\phi)^2 \leq \psi\phi^2 \leq C \left(\frac{1}{R^4} + \frac{1}{R^2} + \frac{1}{T^2} \right).$$

Since $\psi \equiv 1$ on $D_{R/2, T/2}$,

$$\sup_{D_{R/2, T/2}} \frac{|\vec{H}|}{b-\varphi} \leq C \left(\frac{1}{R} + \frac{1}{\sqrt{R}} + \frac{1}{\sqrt{T}} \right).$$

This completes the proof of Theorem 3.1.

4. ENTIRE GRAPHS IN ARBITRARY CODIMENSION WITH FLAT NORMAL BUNDLE

In the last section, we mention arbitrary codimensional case without proof (see [9, 10], [19]). Let us consider an isometric immersion $F_0 : M^n \rightarrow \mathbb{R}^{n+m}$, and its mean curvature flow:

$$\begin{cases} \frac{\partial F}{\partial t}(p, t) = \vec{H}(p, t), & p \in M^n, t \geq 0, \\ F(\cdot, 0) = F_0. \end{cases}$$

Although mean curvature flow in higher codimension is interesting problem, in general, it becomes rather difficult to deal with since the behavior of the normal bundle is complicated. Few results are known in this field except for special cases (for example [1]).

Let $\omega \in \Omega^n(\mathbb{R}^{n+m})$ be a parallel n -form with $|\omega| = 1$. Then ω induces a function w on the immersion $F_0 : M^n \rightarrow \mathbb{R}^{n+m}$ by

$$F_0^* \omega = w d\mu,$$

where $d\mu$ is the induced volume form on M^n . Since $|\omega| = 1$, we must have

$$-1 \leq w(p) \leq 1, \quad \forall p \in M^n.$$

The condition to be a graph then expressed by $w > 0$. Note that there holds

$$(4.1) \quad |\nabla w|^2 \leq m|A|^2.$$

Some people consider mean curvature flow of graphs in higher codimension. There are several long-time existence results (for example [23], [24]). However, there still remain difficulties.

We say that a submanifold has *flat normal bundle* if its normal curvature R^\perp is vanish everywhere. In [19], Smoczyk, Wang and Xin considered mean curvature flow in arbitrary codimension with flat normal bundle, and showed that the flat normal condition is preserved under the mean curvature flow. In this case, we can compute evolution equations similarly to hypersurface cases.

Lemma 4.1. *Entire graphic mean curvature flows with flat normal bundle satisfy the following:*

$$(4.2) \quad (\partial_t - \Delta)w = |A|^2 w,$$

$$(4.3) \quad (\partial_t - \Delta)|A|^2 \leq -2|\nabla^\perp A|^2 + 2|A|^4.$$

Now, we are in the similar situation to Theorem 3.1. The same technique actually works well on the flat normal case and we obtain the following non-existence result.

Theorem 4.2. *Let $(M_t)_{t \in (-\infty, \infty)}$ be an entire graphic eternal solution in arbitrary codimension with flat normal bundle. Suppose that there exist constants $\delta > 0$ and $C < \infty$ so that*

$$w(p, t) \geq \delta > 0, \quad |\vec{H}(p, t)| \leq C < \infty \quad \forall p \in M^n, \forall t \in (-\infty, \infty).$$

Then the eternal solution must be flat for each time slice M_t .

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